Adiabatic renormalization in theories with modified dispersion relations

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Abstract

We generalize the adiabatic renormalization to theories with dispersion relations modified at energies higher than a new scale M_C. We obtain explicit expressions for the mean value of the stress tensor in the adiabatic vacuum, up to the second adiabatic order. We show that for any dispersion relation the divergences can be absorbed into the bare gravitational constants of the theory. We also point out that, depending on the renormalization prescription, the renormalized stress tensor may contain finite trans-Planckian corrections even in the limit $M_C \to \infty$.

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1 Introduction

Inflationary scenarios provide an explanation for the large scale structure of the Universe and for the anisotropy in the Cosmic Microwave Background (CMB). The exponential (or quasi exponential) expansion stretches the physical wavelengths, so that a density fluctuation which is today of cosmological scale was originated during inflation on scales much smaller than the Hubble radius. If the inflationary period lasts enough to solve the causality and other problems, the scales of interest today are not only within the horizon but are also *sub-Planckian* at the beginning of inflation [1]. This fact, known as the *trans-Planckian problem*, provides a potential window to observe consequences of the Planck scale physics.

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Hence the possibility of observing signatures of Planckian physics in the power spectrum of the CMB and in the evolution of the Universe has been widely studied. In the absence of a full quantum theory of gravity, the analysis must be phenomenological. One possibility is to consider an effective field theory approach in which the new physics is encoded in the state of the quantum fields when they leave sub-Planckian scales [2]. Other possibility, which will be analyzed here, is to consider modified dispersion relations for the modes of quantum fields, which might arise in loop quantum gravity or due to the interaction with gravitons [3]. It is important to test the robustness of inflationary predictions under such modifications.

In simple models with a scalar field ϕ, the information on the power spectrum of the CMB is contained in the two point function $\langle \phi(x)\phi(x') \rangle$, and the backreaction of the field on the spacetime metric is contained in the expectation value $\langle T_{\mu\nu} \rangle$. A consistent treatment of the backreaction problem should rely in a careful evaluation of the expectation value of the energy density and pressure, acting as the source in the Semiclassical Einstein Equations (SEE). In general, $\langle T_{\mu\nu} \rangle$ is a divergent quantity. In previous works [4, 5], the renormalization prescription consisted basically in subtracting the ground state energy of each Fourier mode, but this may lead to inconsistencies for quantum fields in curved spaces [6].

The renormalization procedure for quantum fields satisfying the standard dispersion relations in curved backgrounds is well established [6]. The adiabatic renormalization [7], consists in the subtraction of the stress tensor constructed with the WKB expansion of the field modes, up to the fourth adiabatic order. The divergences are proportional to geometric conserved tensors, and can be absorbed into the bare constants of the theory. Thus one defines the renormalized stress tensor as $\langle T^{\mu\nu} \rangle - \langle T^{\mu\nu} \rangle^{(\mathrm{Ad})}$. Here we show that in the case of generalized dispersion relations $\omega^2 \sim k^{2r},\ r \geq 2$, fourth or higher adiabatic order contributions are already finite in $3+1$ dimensions. This suggests that no terms quadratic in the curvature would be necessary in the SEE, and only a redefinition of the cosmological constant and the Newton constant would be required to absorb the divergent contributions. However, as we will see, this naive argument is incorrect: this prescription would lead to a discontinuity in the order of the adiabatic subtraction, and may leave a mark of trans-Planckian physics as a non vanishing contribution to the stress tensor even in the limit $M_C \to \infty$. We will discuss this subtlety using as example the calculation of the trace anomaly in 1+1 dimensions.

2 The WKB expansion

We consider a scalar field ϕ with a non standard dispersion relation induced by higher spatial derivatives

$$\omega_k^2 = k^2 + C(\eta)\left[m^2 + 2\sum_{s,p}(-1)^{s+p}\, b_{sp}\, \left(\frac{k}{C^{1/2}(\eta)}\right)^{2(s+p)}\right], \tag{1}$$

where b_{sp} are arbitrary coefficients of order $M_C^{2(1-s-p)}$, and $p \leq s$. We work with a general spatially flat FRW metric given by $ds^2 = C(\eta)[-d\eta^2 + \delta_{ij}dx^i dx^j]$ where $C^{1/2}(\eta)$ is the scale factor given as a function of the conformal time η.

The Fourier modes χ_k corresponding to the scaled field $\chi = C^{(n-2)/4}\phi$ satisfy

$$\chi_k'' + \left[(\xi - \xi_n)RC + \omega_k^2\right]\chi_k = 0\,. \tag{2}$$

Here primes stand for derivatives with respect to the conformal time η, R is the Ricci scalar, and ξ defines the coupling with the curvature. The field modes χ_k can be expressed in the well known form

$$\chi_k = \frac{1}{\sqrt{2W_k}}\exp\left(-i\int^{\eta} W_k(\tilde{\eta})d\tilde{\eta}\right). \tag{3}$$

Substitution of Eq. (3) into Eq. (2) yields a nonlinear differential equation for W_k that can be solved iteratively by assuming that W_k is a slowly varying function of η. In this WKB approximation, the adiabatic order of a given term is defined as the number of derivatives of the metric. Working up to the second adiabatic order, we straightforwardly obtain [8]

$$\begin{aligned} W_k^2 &= \omega_k^2 + (\xi - \xi_n)\,(n-1)\left(\frac{C''}{C} + \frac{(n-6)}{4}\frac{C'^2}{C^2}\right) - \frac{1}{4}\frac{C''}{C}\left(1 - \frac{k^2}{\omega_k^2}\frac{d\omega_k^2}{dk^2}\right) \\ &- \frac{1}{4}\frac{C'^2}{C^2}\frac{k^4}{\omega_k^2}\frac{d^2\omega_k^2}{d(k^2)^2} + \frac{5}{16}\frac{C'^2}{C^2}\left(1 - \frac{k^2}{\omega_k^2}\frac{d\omega_k^2}{dk^2}\right)^2. \end{aligned} \tag{4}$$

From Eq. (4) it is clear that while the zeroth adiabatic order scales as ω_k^2, the second order scales as ω_k^0. It can be shown that the $2j-$adiabatic order scales as ω_k^{2-2j}.

3 Renormalization of the stress tensor

We start from the vacuum expectation values of the energy density ρ and pressure p, generalized to arbitrary dimension n and coupling ξ [8]:

$$
\begin{aligned}
\langle\rho\rangle &= \frac{1}{\sqrt{C}}\int \frac{d^{n-1}k\,\mu^{\bar{n}-n}}{(2\pi\sqrt{C})^{(n-1)}}\left\{\frac{C^{(n-2)/2}}{2}\left|\left(\frac{\chi_k}{C^{(n-2)/4}}\right)'\right|^2+\xi G_{\eta\eta}|\chi_k|^2\right.\\
&+ \left.\frac{\omega_k^2}{2}|\chi_k|^2+\xi\frac{(n-1)}{2}\left[\frac{C'}{C}(\chi_k'\chi_k^*+\chi_k\chi_k'^*)-\frac{C'^2}{C^2}\frac{(n-2)}{2}|\chi_k|^2\right]\right\}, \qquad (5)\\
\langle p\rangle &= \frac{1}{\sqrt{C}}\int \frac{d^{n-1}k\,\mu^{\bar{n}-n}}{(2\pi\sqrt{C})^{(n-1)}}\left\{\left(\frac{1}{2}-2\xi\right)C^{(n-2)/2}\left|\left(\frac{\chi_k}{C^{(n-2)/4}}\right)'\right|^2\right.\\
&+ \xi G_{11}|\chi_k|^2+\left[\left(\frac{k^2}{n-1}\right)\frac{d\omega_k^2}{dk^2}-\frac{\omega_k^2}{2}\right]|\chi_k|^2-\xi(\chi_k''\chi_k^*+\chi_k\chi_k''^*)\\
&+ \left.\xi\frac{C'}{2C}(\chi_k'\chi_k^*+\chi_k\chi_k'^*)-\xi\frac{(n-2)}{2}\left(\frac{C''}{C}-\frac{(8-n)}{4}\frac{C'^2}{C^2}\right)|\chi_k|^2\right\}. \qquad (6)
\end{aligned}
$$

Here $\bar{n}$ is the number of dimensions of the physical spacetime, μ is an arbitrary parameter with mass dimension and $G_{\eta\eta}$ and $G_{11}(=G_{22}=...=G_{\bar{n}\bar{n}})$ are the nontrivial components of the Einstein tensor.

The vacuum expectation values $\langle\rho\rangle$ and $\langle p\rangle$ are found from Eqs. (3), (5) and (6). Knowing the dependence with k of the $2j-$adiabatic order one can show that, for $\omega_k^2 \sim k^{2r}$ with $r \geq 4$, all contributions of second or higher adiabatic order are finite. The divergences come only from the zeroth order terms contained in $\langle\rho\rangle$ and $\langle p\rangle$. Instead, in the cases $\omega_k^2 \sim k^6$ and $\omega_k^2 \sim k^4$, though no fourth order divergences appear, second order terms include divergent contributions in $3+1$ dimensions.

The zeroth and second adiabatic orders of the stress tensor can be computed from Eqs. (3) to (6). After a long calculation we obtain [8]

$$
\langle T_{\mu\nu}\rangle^{(0)} = -\frac{g_{\mu\nu}}{4}\frac{\Omega_{n-1}\,\mu^{\bar{n}-n}}{(2\pi)^{n-1}}I_1\,, \qquad (7)
$$

$$
\langle T_{\mu\nu}\rangle^{(2)} = G_{\mu\nu}\frac{\Omega_{n-1}\,\mu^{\bar{n}-n}}{4\,(2\pi)^{n-1}}\left\{\frac{I_2}{6(n-1)(n-2)}+(\xi-\frac{1}{6})I_1\right\}, \qquad (8)
$$

$$
I_1 = \int_0^\infty dx\,\frac{x^{(n-3)/2}}{\tilde{\omega}_k}, \qquad I_2 = \int_0^\infty dx\,\frac{x^{(n+1)/2}}{\tilde{\omega}_k^3}\frac{d^2\tilde{\omega}_k^2}{dx^2}, \qquad (9)
$$

where $\Omega_{n-1} \equiv 2\pi^{(n-1)/2}/\Gamma[(n-1)/2]$, $x \equiv k^2/C$ and $\tilde{\omega}_k = \omega_k/\sqrt{C}$. To get these results, we performed several integrations by parts, and used the fact that in dimensional regularization the integral of a total derivative vanishes. For the usual dispersion relation, the integral I_2 vanishes and we recover the known second adiabatic order results [7].

Eqs. (7), (8) and (9) show that $\langle T_{\mu\nu}\rangle^{(0)} = N_0 g_{\mu\nu}$ and $\langle T_{\mu\nu}\rangle^{(2)} = N_2 G_{\mu\nu}$, where N_0 and N_2 are divergent factors in $3+1$ dimensions. Hence these contributions can be absorbed into a renormalization of the cosmological and Newton constants: we can define $\langle T_{\mu\nu}\rangle_{\rm Ren} = \langle T_{\mu\nu}\rangle - \langle T_{\mu\nu}\rangle^{(0)} - \langle T_{\mu\nu}\rangle^{(2)}$ and write the SEE as

$$G_{\mu\nu} + \Lambda_{\rm R} g_{\mu\nu} = 8\pi G_{\rm R} \langle T_{\mu\nu}\rangle_{\rm Ren}. \tag{10}$$

Differing from the standard case $\omega_k^2 \sim k^2$, all contributions of adiabatic orders higher than the second are finite, which suggests that no terms quadratic in the curvature would be necessary in the SEE.

4 A discontinuity in the limit $M_C \to \infty$?

Assuming that the usual dispersion relation is valid for all energies, the renormalization of the stress tensor in 3+1 dimensions is achieved by subtracting the fourth adiabatic order [6, 7]. This procedure is equivalent to a redefinition of the bare constants in the effective Lagrangian, and is independent of the particular metric considered. It is necessary to renormalize not only the cosmological and Newton constants, but also to include in the gravitational Lagrangian the three quadratic terms $\alpha_0 R^2 + \beta_0 R_{\mu\nu}R^{\mu\nu} + \gamma_0 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, and absorb the infinities into the bare constants α_0, β_0 and γ_0. The conformal anomaly is a well known consequence of this renormalization scheme. However, if at some very high energy scale M_C the dispersion relation gets modified by a term of the form $k^{2r}/M_C^{2(r-1)}$, $r \geq 2$, only the second adiabatic order is divergent and, as described above, it would be enough to "dress" the cosmological and Newton constants. This "discontinuity" in the order of the subtraction may produce non vanishing trans-Planckian contributions to the renormalized stress tensor, even in the limit $M_C \to \infty$.

However, there is a subtle point in this argument. While by power counting the fourth adiabatic order is finite, divergences may appear when $\langle T_{\mu\nu}\rangle^{(4)}$ is expressed in terms of geometric tensors in n dimensions. Indeed, this is the case in lower dimensions, as we will illustrate in what follows. Let us consider a conformally coupled ($\xi = 0$) massless field in $1+1$ dimensions, with a dispersion relation of the form $\omega^2 = k^2 + 2bk^4/C(\eta)$. For FRW metrics in n dimensions $G_{\mu\nu}$ is proportional to $n-2$ (this tensor vanishes as $n \to 2$, because it is the variation of the would be Gauss-Bonnet topological invariant at $n = 2$). Therefore, from Eqs. (7), (8) and (9) one readily sees that only the zeroth adiabatic order is divergent in the limit $n \to 2$. One would naively conclude that the subtraction of this order would be enough, even for the usual dispersion relation, where it is known that it is necessary to subtract up to the second adiabatic order [6]. The point is that while the geometric tensor

$G_{\mu\nu}$ vanishes in exactly two dimensions, in Eq. (8) it is multiplied by a function that has a simple pole at $n = 2$. Therefore, as the poles must be absorbed into the bare constants before taking the limit $n \to 2$, it is necessary to subtract up to the second adiabatic order, whatever the dispersion relation. The renormalized trace of the stress tensor is then defined as

$$\langle T \rangle_{\rm Ren} = \langle T \rangle_{\rm modes} - \langle T \rangle^{(0)} - \langle T \rangle^{(2)} \tag{11}$$

where $\langle T \rangle_{\rm modes}$ is the unrenormalized trace computed using Eq. (6) and the explicit expressions for the modes of the field. The case $b = 0$ is very well known: $\langle T \rangle_{\rm modes}$ vanishes due to conformal invariance, $\langle T \rangle^{(0)}$ also vanishes and one recovers the usual trace anomaly $\langle T \rangle_{\rm Ren} = -\langle T \rangle^{(2)} = R/24\pi$.

Let us now assume that $b \neq 0$, and compute the renormalized trace in de Sitter space $C(\eta) = \alpha^2/\eta^2$. We have

$$\langle T \rangle_{\rm modes} - \langle T \rangle^{(0)} = \frac{1}{\pi C} \int_0^\infty dk \left(1 - \frac{k^2}{\omega_k^2} \frac{d\omega_k^2}{dk^2} \right) \left(\omega_k^2 |\chi_k|^2 - \frac{\omega_k}{2} \right) \tag{12}$$

The modes of the field satisfy

$$|\chi_k|^2 = \frac{e^{-\lambda\pi/4}}{k} \sqrt{\frac{\lambda}{2}} |D_{-(1+i\lambda)/2}[(i+1)s]|^2 \equiv \frac{1}{k} f(\lambda, s), \tag{13}$$

where D is the parabolic function, $s = k\eta/\sqrt{\lambda}$ and $\lambda = \alpha/\sqrt{2b}$. After changing variables and some algebra we get

$$\langle T \rangle_{\rm modes} - \langle T \rangle^{(0)} = \frac{R}{2\pi} \int_0^\infty ds s^3 \left\{ f(\lambda, s) - \frac{\sqrt{\lambda}}{2\sqrt{\lambda + s^2}} \right\} \tag{14}$$

A numerical evaluation of the integral gives, in the limit $M_C \to \infty$, $\langle T \rangle_{\rm modes} - \langle T \rangle^{(0)} = -R/24\pi$. As for the case of the usual dispersion relation $b = 0$, the trace of the stress tensor has an anomaly. However, the numerical value does not coincide with the usual one (it differs by a sign). Therefore, if we subtract only the zeroth adiabatic order, there is a discontinuity in the renormalized stress tensor as $M_C \to \infty$. This discontinuity disappears if, as already mentioned, we also subtract the second adiabatic order. Indeed, from Eq. (8) we find, near $n = 2$,

$$\langle T \rangle^{(2)} = -\frac{R}{48\pi} \left((n-2) I_1 + I_2 \right) \mu^{2-n} \tag{15}$$

As I_1 is finite for nonvanishing b, the first term does not contribute to the trace in $n = 2$. On the other hand, an explicit evaluation of I_2 gives, in the limit $b \to 0$, $\langle T \rangle^{(2)} = -R/12\pi$. Therefore, combining Eqs. (11), (14) and (15) we see that the usual trace anomaly is recovered in the limit $M_C \to \infty$.

5 Conclusions

The adiabatic subtraction can be generalized to theories with modified dispersion relations: for any dispersion relation, the zeroth and second adiabatic orders of the stress tensor are proportional to $g_{\mu\nu}$ and $G_{\mu\nu}$ respectively. In $3+1$ dimensions, the higher powers of k^2 in the dispersion relation make finite the fourth adiabatic order. Therefore, in order to get a finite mean value of the stress tensor, it would be enough to subtract up to the second adiabatic order. This would be possible for any value of the new physics scale M_C. However, as we have shown with a simple example in $1+1$ dimensions, this renormalization prescription, which is not equivalent to a redefinition of the bare constants in the effective Lagrangian of the theory, would lead to a discontinuity in the stress tensor as $M_C \to \infty$, and to a wrong value of the trace anomaly. It is likely that the finite fourth adiabatic order near 3+1 dimensions, when written in terms of n-dimensional geometric tensors, will also have poles. This may happen for a term proportional to the variation of the would be Gauss-Bonnet topological invariant near $n = 4$. In this situation, a consistent renormalization would involve the subtraction of the fourth adiabatic order. Work in this direction is in progress.

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